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ALGEBRA.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicago, Ill.

Prove the identities

$$2-\sqrt[1]{2}=\frac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\dots\dots$$

$$\frac{5-\sqrt[1]{5}}{2}=\frac{1}{1}+\frac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\dots\dots$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let N be any number, then

$$N=\frac{Nr^m}{r^m}=\frac{r^m}{r^m}=\frac{p^m}{r^m}\left[1-\left(\frac{p^m-Nr^m}{p^m}\right)\right].$$

$$\begin{aligned}\therefore \sqrt[m]{N} &= \frac{p}{r} \left[1 - \frac{p^m - Nr^m}{p^m}\right]^{1/m} = \frac{p}{r} \left[1 - \left(\frac{p^m - Nr^m}{mp^m}\right)\right. \\ &\quad - \frac{(m-1)}{1.2} \left(\frac{p^m - Nr^m}{mp^m}\right)^2 - \frac{(m-1)(2m-1)}{1.2.3} \left(\frac{p^m - Nr^m}{mp^m}\right)^3 \\ &\quad \left. - \frac{(m-1)(2m-1)(3m-1)}{1.2.3.4} \left(\frac{p^m - Nr^m}{mp^m}\right)^4 - \text{etc.}\right].\end{aligned}$$

Let $m=2$ and $p^m - Nr^m = 1$,

$$\begin{aligned}\therefore \sqrt[1]{N} &= \frac{p}{r} \left(1 - \frac{1}{p^2}\right)^{\frac{1}{2}} = \frac{p}{r} \left[1 - \frac{1}{2} \left(\frac{1}{p^2}\right) - \frac{1}{2^3} \left(\frac{1}{p^2}\right)^2 - \frac{1}{2^4} \left(\frac{1}{p^2}\right)^3\right. \\ &\quad \left. - \frac{5}{2^5} \left(\frac{1}{p^2}\right)^4 - \text{etc.}\right] \dots\dots\dots (1).\end{aligned}$$

Let $m=2$ and $p^m - Nr^m = 4$,

$$\begin{aligned}\therefore \sqrt[1]{N} &= \frac{p}{r} \left(1 - \frac{4}{p^2}\right)^{\frac{1}{2}} = \frac{p}{r} \left[1 - \frac{1}{2} \left(\frac{4}{p^2}\right) - \frac{1}{1.2.2^2} \left(\frac{4}{p^2}\right)^2\right. \\ &\quad \left. - \frac{1.3}{1.2.3.2^3} \left(\frac{4}{p^2}\right)^3 - \frac{1.3.5}{1.2.3.4.2^4} \left(\frac{4}{p^2}\right)^4 - \text{etc.}\right].\end{aligned}$$

$$\therefore \sqrt[1]{N} = \frac{p}{r} \left[1 - \frac{2}{p^2} - \frac{2}{p^4} - \frac{4}{p^6} - \frac{2.5}{p^8} - \frac{4.7}{p^{10}} - \frac{3.4.7}{p^{12}} - \text{etc.}\right] \dots(2).$$

In (1) let $N=2$, $p=17$, $r=12$, $\therefore p^2 - 2r^2 = 1$.

$$\therefore \sqrt[1]{2} = \frac{1}{12} \left(1 - \frac{1}{2.17^2} - \frac{1}{2^3.17^4} - \frac{1}{2^4.17^6} - \frac{5}{2^5.17^8} - \text{etc.}\right).$$

$$\therefore 2-\sqrt{2}=2-1\frac{1}{2}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{etc.}$$

$$2-\sqrt{2}=1\frac{1}{2}+1\frac{1}{3}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{etc.}$$

$$\begin{aligned} &=1\frac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\left(\frac{1}{2^4.3.17.577}-\frac{1}{2^4.3.17.577.578}\right) \\ &\quad +\left(\frac{1}{2^4.3.17.577.578}-\frac{1}{2^4.3.17.577.578^2}\right) \\ &\quad +\left(\frac{1}{2^4.3.17.577.578^2}+\frac{1}{2^5.3.17.577.665857}-\right)+\text{etc.} \end{aligned}$$

$$\therefore 2-\sqrt{2}=1\frac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}+\frac{1}{2^5.3.17.577.665857}+\dots\dots$$

In (2) let $N=5$, $p=7$, $r=3$, $\therefore p^2-5r^2=4$.

$$\therefore \sqrt{5}=\frac{7}{3}\left(1-\frac{2}{7^2}-\frac{2}{7^4}-\frac{4}{7^6}-\frac{2.5}{7^8}-\frac{4.7}{7^{10}}-\text{etc.}\right)$$

$$\frac{5}{\sqrt{2}}=\frac{6}{7}-\frac{1}{3.7}-\frac{1}{3.7^3}-\frac{2}{3.7^5}-\frac{5}{3.7^7}-\frac{2}{3.7^9}-\text{etc.}$$

$$\frac{5-\sqrt{5}}{2}=\frac{5}{2}-\frac{1}{6}+\frac{1}{3.7}+\frac{1}{3.7^3}+\frac{2}{3.7^5}+\frac{5}{3.7^7}+\frac{2}{3.7^9}+\text{etc.}=1+\frac{1}{3}+\frac{1}{3.7}$$

$$\begin{aligned} &+\left(\frac{1}{3.7.47}-\frac{2}{3.7.47.49}\right)+\left(\frac{2}{3.7.47.49}-\frac{4}{3.7.47.49^2}\right) \\ &\quad +\left(\frac{4}{3.7.47.49^2}+\frac{1}{3.7.47.2207}-\right)+\text{etc.} \end{aligned}$$

$$\therefore \frac{5-\sqrt{5}}{2}=1+\frac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}+\dots\dots$$

The above formulæ were used by Dr. Artemas Martin for extracting the root of numbers several years ago.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $2-\sqrt{2}$ is rather more than $\frac{1}{3}$, we can, confining ourselves to 4 terms, put $2-\sqrt{2}=1\frac{1}{2}+\frac{1}{2^2x}+\frac{1}{2^3xy}+\frac{1}{2^4xyz}$, or by transposition, $\sqrt{2}=\frac{3}{2}-\frac{1}{4x}-\frac{1}{8xy}-\frac{1}{16xyz}$. Squaring and omitting the term $\frac{1}{256x^2y^2z^2}$, we get, after multiplying by 64 x ,

$$16x=48-\frac{4}{x}+\frac{24}{y}-\frac{1}{xy^2}+\frac{12}{yz}-\frac{4}{xy}-\frac{2}{xyz}-\frac{1}{xy^2z},$$

whence $x=3$, and substituting this value and multiplying by $3y$, we get

$$4y=68-\frac{1}{y}+\frac{34}{z}-\frac{1}{yz};$$

$\therefore y=17$, and substituting it, and multiplying by $17z$, we get $z=578-1=577$. If another term had been desired we would have annexed to the above series the term $\frac{1}{2^5xyz}$ and proceeded in the same manner, though for 4 terms, the series is sufficiently convergent to render the value of $2-\sqrt{2}$ correct for at least 8 decimals.

Since $\frac{5-\sqrt{5}}{2}$ is $=1 + \text{fraction}$, we put, restricting the series to 4 terms besides 1,

$$\frac{5-\sqrt{5}}{2}=1+\frac{1}{x}+\frac{1}{xy}+\frac{1}{xyz}+\frac{1}{xyzt},$$

$$\text{whence } \sqrt{5}=3-\frac{2}{x}-\frac{2}{xy}-\frac{2}{xyz}-\frac{2}{xyzt}.$$

Squaring and omitting the term $4/x^2y^2z^2t^2$, we have, transposing, suppressing the common factor 4, and multiplying by x ,

$$x=3-1/x+3/y-1/xy^2-1/xy^2z^2+3/yz+3/yzt-2/xy-2/xyz-2/xyzt-2/xy^2z-2/xy^2z^2t, \text{ whence } x=3.$$

Substituting this value, and multiplying by $3y$, we have

$$y=7-1/y-1/yz^2+7/z+7/zt-2/yz-2/yz^2t; \text{ whence } y=7.$$

Substituting, and multiplying by $7z$, we get

$$z=47-1/z+47/t-2/zt; \text{ whence } z=47.$$

Substituting, and multiplying by $47t$, we get $t=2207$.

Such series converge very rapidly. In German works they are called "Theilbruchreihen," signifying *partial fraction series*. Every common fraction may be converted into such a series by the following process.

Let the fraction be $\frac{1}{15}$.

$$15)19(2$$

$$\therefore \frac{1}{15}=\frac{1}{2}+1/2.2+1/2.2.7+1/2.2.7.10+1/2.2.7.10.19, \quad 11)19(2$$

so that, taking successively 1, 2, 3, 4 of these, we get as in continuous fractions, approximate values, as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{11}{14}$, $\frac{22}{27}$. As in continuous fractions, the numerator of the difference of any two consecutive approximate fractions is always $=1$. Supposing the indeterminate equation $15x-19y=1$. Changing $\frac{1}{15}$ by the above method into such a

$$3)19(7$$

$$2)19(10$$

$$1)19(19$$

series, and take the last approximate fraction, viz, $\frac{221}{14}$, $x=280$, $y=221$, will furnish two values; also, the preceding fraction $\frac{11}{1}$, $x=14$, $y=11$.

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

Solve the equation, $(6x^2 + x - 3)^2 - 48^2 = (x + 15)^2$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and A. H. BELL, Hillsboro, Ill.

$$(6x^2 + x - 3)^2 - (48)^2 = (x + 15)^2.$$

$$\therefore 3x^4 + x^3 - 3x^2 - 3x - 210 = 0.$$

$$\therefore x_1 = 2.924412 +,$$

$$x_2 = -3.041623 -,$$

$$x_3 = 2.804395\sqrt{-1} - .158044.$$

$$x_4 = -2.804395\sqrt{-1} - .158044.$$

II. Solution by J. MARCUS BOORMAN, Consultative Mechanician, Counsellor at Law, Inventor, Etc., Hewlett, Long Island, New York.

The real roots of $x^4 + \frac{1}{3}x^3 - x^2 - x = 70$ (the forms to which the given equation reduces) are :

$$\begin{array}{l} x = +2.924412 \ 149966 \ 623189 \ 6108(58211) \\ x_1 = -3.041622 \ 694570 \ 750484 \ 61819(75892) \end{array} \left. \vphantom{\begin{array}{l} x \\ x_1 \end{array}} \right\} \begin{array}{l} \text{true to "('') marks,—probably} \\ \text{25 decimals true.} \end{array}$$

Found thus: $x^4 +$	$\frac{1}{3}x^3$	$-1x^2$	$-1x$	-70
At sight $x = \pm 3$ (near), try -3.04		$+8.24-$	-22.01	69.9504
	<hr style="width: 100px; margin: 0;"/> -2.71- $\times 3$	<hr style="width: 100px; margin: 0;"/> +7.24-	<hr style="width: 100px; margin: 0;"/> -23.01	<hr style="width: 100px; margin: 0;"/> 0.04-
Multiply by -3 , $-.04 + .13$		<hr style="width: 100px; margin: 0;"/> 21.72	<hr style="width: 100px; margin: 0;"/> 69.03	<hr style="width: 100px; margin: 0;"/> error +*
	<hr style="width: 100px; margin: 0;"/> .1084+	<hr style="width: 100px; margin: 0;"/> 29-	<hr style="width: 100px; margin: 0;"/> .9204	
	<hr style="width: 100px; margin: 0;"/> +8.24-	<hr style="width: 100px; margin: 0;"/> -22.01	<hr style="width: 100px; margin: 0;"/> 69.9504	

*The root \therefore is $-2.0316 +$ nearly.

Like treatment by $+2.9$ gives 2.925 near true. I get the rest in a more concise way (shorter than Horner's).

The resulting equations of above roots are : (the coefficients appear on face of computation)

$$0 = x^3 + (+\text{root} + \frac{1}{3})x^2 + 8.526990 \ 472861 \ 28178 \text{etc. } x + 23.936434, 541435, 17396 +.$$

$$0 = x^3 + (-\text{root} + \frac{1}{3})x^2 + 7.237594 \ 384604 \ 24939 \text{etc. } x - 23.914031, 334310, 10968 +.$$

$$0 = (5.966034, 84 \text{etc})x^2 + (1.289496, 088257, 03238 +)x + 46.950465, 875795, 28364 +.$$

Hence $x^2 + 0.216122, 788722x = -7.869626, 49385$ etc.

$$\therefore x_1 = -0.108 \text{ etc. } \pm \sqrt{-7.890 +}.$$

Shorter and more accurate than Horner's method, especially for 5th degree and 6th degree equations.

The other two roots are imaginary.